

# Coloring graphs with no even hole $\geq 6$ : the triangle-free case

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## Abstract

In this paper, we prove that the class of graphs with no triangle and no induced cycle of even length at least 6 has bounded chromatic number. It is well-known that even-hole-free graphs are  $\chi$ -bounded but we allow here the existence of  $C_4$ . The proof relies on the concept of Parity Changing Path, an adaptation of Trinity Changing Path which was recently introduced by Bonamy, Charbit and Thomassé to prove that graphs with no induced cycle of length divisible by three have bounded chromatic number.

## 1 Introduction

A *hole* in a graph is an induced cycle of length at least four. A *proper coloring* of a graph is a function that assigns to each vertex a color with the constraint that two adjacent vertices are not given the same color. The *chromatic number* of  $G$ , denoted by  $\chi(G)$ , is the smallest number of colors needed to color the graph properly. All the colorings considered in the sequel are proper, so we just call them colorings. The size of the largest clique of  $G$  is denoted  $\omega(G)$ . We obviously have  $\omega(G) \leq \chi(G)$ , and one may wonder whether the equality holds. In fact, it does not hold in the general case, and the simplest counter-example is any *odd hole*, i.e. any hole of odd length, for which  $\omega(G) = 2$  but  $\chi(G) = 3$ . Graphs for which the equality  $\chi(G') = \omega(G')$  holds for every induced subgraph  $G'$  of  $G$  are called *perfect*, and the Strong Perfect Graph Theorem [4] proved that a graph is perfect if and only if it is *Berge*, that is to say there is no odd hole in  $G$  nor in its complement. In order to get some upper bound on  $\chi(G)$ , Gyárfás [11] introduced the concept of  $\chi$ -bounded class: a family  $\mathcal{G}$  of graphs is called  *$\chi$ -bounded* if there exists a function  $f$  such that  $\chi(G') \leq f(\omega(G'))$  whenever  $G'$  is an induced subgraph of  $G \in \mathcal{G}$ .

This notion has widely been studied since then, in particular in hereditary classes (*hereditary* means closed under taking induced subgraph). A classical result of Erdős [9] asserts that there exist graphs with arbitrarily large *girth* (that is, the length of the shortest induced cycle) and arbitrarily large chromatic number. Thus forbidding only one induced subgraph  $H$  may lead to a  $\chi$ -bounded class only if  $H$  is acyclic. It is conjectured that this condition is also sufficient [10, 22], but it is proved only if  $H$  is a path, a star [11] or a tree of radius two [13] (or three, with additional conditions [14]). Scott [19] also proved it for any tree  $H$ , provided that we forbid every induced subdivision of  $H$ , instead of just  $H$  itself.

Consequently, forbidding holes in order to get a  $\chi$ -bounded class is conceivable only if we forbid infinitely many hole lengths. Two parameters should be taken into account: first, the length of the holes, and secondly, the parity of their lengths. In this respect, Gyárfás [11] made

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a famous series of three conjectures. The first one asserts the class of graphs with no odd hole is  $\chi$ -bounded. The second one asserts that, for every  $k$ , the class of graphs with no hole of length at least  $k$  is  $\chi$ -bounded. The last one generalizes the first two conjectures and asserts that for every  $k$ , the class of graphs with no odd hole of length at least  $k$  is  $\chi$ -bounded. After several partial results [17, 20, 5], the first and the second conjectures were recently solved by Chudnovsky, Scott and Seymour [21, 6]. Moreover, we learned while writing this article that Scott and Seymour claimed<sup>1</sup> to have proved a very general result implying the triangle-free case of the third conjecture (which would also imply the result of this paper): for every  $k \geq 0$ , every triangle-free graph with large enough chromatic number admits a sequence of holes of  $k$  consecutive lengths.

The class of even-hole-free graphs has extensively been studied from a structural point of view. A decomposition theorem together with a recognition algorithm have been found by Conforti, Cornuéjols, Kapoor and Vušković [7, 8, 3]. Reed conjectured [12] that every even-hole-free graph has a vertex whose neighborhood is the union of two cliques (called a *bisimplicial* vertex), which he and his co-authors proved [1] a few years later. As a consequence, they obtained that every even-hole-free graph  $G$  satisfies  $\chi(G) \leq 2\omega(G) - 1$ .

Forbidding  $C_4$  is in fact a strong restriction since  $C_4$  can also be seen as the complete bipartite graph  $K_{2,2}$ : Kühn and Osthus [15] proved that for every graph  $H$  and for every integer  $s$ , every graph of large average degree (with respect to  $H$  and  $s$ ) with no  $K_{s,s}$  as a (non-necessarily induced) subgraph contains an induced subdivision of  $H$ , where each edge is subdivided at least once. This strong result implies that  $\chi$  is bounded in any class  $\mathcal{C}$  defined as graphs with no triangles, no induced  $C_4$  and no cycles of length divisible by  $k$ , for any fixed integer  $k$ . Indeed, let  $G \in \mathcal{C}$  be a minimal counter-example to  $\chi(G) \leq t$  (with  $t$  chosen large enough with respect to  $k$ ), then it has large minimum degree. Moreover it has neither induced  $C_4$  nor triangles, consequently it has no  $C_4$  subgraphs. By Kühn and Osthus' theorem, there exists an induced subdivision  $H$  of  $K_\ell$  for some well-chosen integer  $\ell$  depending on  $k$ . Consider  $K_\ell$  as an auxiliary graph where we color each edge with  $c \in \{1, \dots, k\}$  if this edge is subdivided  $c$  times modulo  $k$  in  $H$ . By Ramsey's theorem [16], if  $\ell$  is large enough, then we can find a monochromatic clique  $K$  of size  $k$ . Let  $C_0$  be a Hamiltonian cycle through  $K$  and call  $C$  the corresponding cycle in the subdivided edges in  $H$ . Since  $K$  was monochromatic in  $K_\ell$ , the edges used in  $C_0$  are subdivided the same number of times modulo  $k$ , consequently  $C$  has length divisible by  $k$ . Moreover, it is an induced cycle since each edge is subdivided at least once in  $H$ .

This is why we are interested in finding a  $\chi$ -boundedness result when every even hole except  $C_4$  is forbidden, which was conjectured by Bruce Reed [18]. In this paper, we achieve a partial result by forbidding also triangles. This is a classical step towards  $\chi$ -boundedness, and Thomassé *et al.* [23] even asked whether this could always be sufficient, namely: does there exist a function  $f$  such that for every class  $\mathcal{C}$  of graphs and any  $G \in \mathcal{C}$ ,  $\chi(G) \leq f(\chi_T(G), \omega(G))$ , where  $\chi_T(G)$  denotes the maximum chromatic number of a triangle-free induced subgraph of  $G$ ?

The result of this paper is closely related to the following recent one, by Bonamy, Charbit and Thomassé, answering to a question by Kalai and Meshulam on the sum of Betti numbers of the stable set complex (see [2] for more details):

**Theorem 1** ([2]). *There exists a constant  $c$  such that every graph  $G$  with no induced cycle of length divisible by 3 satisfies  $\chi(G) < c$ .*

Indeed, the so-called Parity Changing Path (to be defined below) is directly inspired by their Trinity Changing Path. The structure of the proofs also have several similarities.

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<sup>1</sup>Their article is still in preparation.

**Contribution** We prove the following theorem:

**Theorem 2.** *There exists a constant  $c$  such that every graph  $G$  with no triangle and no induced cycle of even length at least 6 satisfies  $\chi(G) < c$ .*

The outline is to prove the result when the 5-hole is also forbidden (see Lemma 3 below), which should intuitively be easier, and then deduce the theorem for the general case.

To begin with, let us introduce and recall some notations: the class under study, namely graphs with no triangle and no induced  $C_{2k}$  with  $k \geq 3$  (meaning that every even hole is forbidden except  $C_4$ ) will be called  $\mathcal{C}_{3,2k \geq 6}$  for short. Moreover, we will consider in Subsection 2 the subclass  $\mathcal{C}_{3,5,2k \geq 6}$  of  $\mathcal{C}_{3,2k \geq 6}$  in which the 5-hole is also forbidden. For two subsets of vertices  $A, B \subseteq V$ ,  $A$  *dominates*  $B$  if  $B \subseteq N(A)$ . A *major connected component* of  $G$  is a connected component  $C$  of  $G$  for which  $\chi(C) = \chi(G)$ . Note that such a component always exists. For any induced path  $P = x_1x_2 \cdots x_\ell$  we say that  $P$  is a path from its *origin*  $x_1$  to its *end*  $x_\ell$  or an  $x_1x_\ell$ -*path*. Its *interior* is  $\{x_2, \dots, x_{\ell-1}\}$  and its *length* is  $\ell - 1$ .

Moreover, we use a rather common technique called a *levelling* [21, 5] : given a vertex  $v$ , the  $v$ -*levelling* is the partition  $(N_0, N_1, \dots, N_k, \dots)$  of the vertices according to their distance to  $v$ :  $N_k$  is the set of vertices at distance exactly  $k$  from  $v$  and is called the  $k$ -*th level*. In particular,  $N_0 = \{v\}$  and  $N_1 = N(v)$ . We need two more facts about levellings: if  $x$  and  $y$  are in the same part  $N_k$  of a  $v$ -levelling, we call an *upper  $xy$ -path* any shortest path from  $x$  to  $y$  among those with interior in  $N_0 \cup \dots \cup N_{k-1}$ . Observe that it always exists since there is an  $xv$ -path and a  $vy$ -path (but it may take shortcuts; in particular, it may be just one edge). Moreover, in any  $v$ -levelling, there exists  $k$  such that  $\chi(N_k) \geq \chi(G)/2$ : indeed, if  $t$  is the maximum of  $\chi(N_i)$  over all levels  $N_i$ , one can color  $G$  using  $2t$  colors by coloring  $G[N_i]$  with the set of colors  $\{1, \dots, t\}$  if  $i$  is odd, and with the set of colors  $\{t+1, \dots, 2t\}$  if  $i$  is even. Such a level with chromatic number at least  $\chi(G)/2$  is called a *colorful level*.

Let us now introduce the main tool of the proof, called *Parity Changing Path* (PCP for short) which, as already mentioned, is inspired by the *Trinity Changing Path* (TCP) appearing in [2]: intuitively (see Figure 1 for an unformal diagram), a PCP is an induced sequence of induced subgraphs and paths  $(G_1, P_1, \dots, G_\ell, P_\ell, H)$  such that each block  $G_i$  can be crossed by two possible paths of different parities, and the last block  $H$  typically is a stock of big chromatic number. Formally, a *PCP of order  $\ell$*  in  $G$  is a sequence of induced subgraphs  $G_1, \dots, G_\ell, H$  (called *blocks*; the  $G_i$  are the *regular blocks*) and induced paths  $P_1, \dots, P_\ell$  such that the origin of  $P_i$  is some vertex  $y_i$  in  $G_i$ , and the end of  $P_i$  is some vertex  $x_{i+1}$  of  $G_{i+1}$  (or of  $H$  if  $i = \ell$ ). Apart from these special vertices which belong to exactly two subgraphs of the PCP, the blocks and paths  $G_1, \dots, G_\ell, H, P_1, \dots, P_\ell$  composing the PCP are pairwise disjoint. The only possible edges have both endpoints belonging to the same block or path. We also have one extra vertex  $x_1 \in G_1$  called the *origin* of the PCP. Moreover in each block  $G_i$ , there exists one induced  $x_iy_i$ -path of odd length, and one induced  $x_iy_i$ -path of even length (these paths are not required to be disjoint one from each other). In particular  $x_i \neq y_i$  and  $x_iy_i$  is not an edge. For technical reasons that will appear later, we also require that  $H$  is connected, every  $G_i$  has chromatic number at most 4 and every  $P_i$  has length at least 2. Finally the chromatic number of  $H$  is called the *leftovers*.

In fact in Subsection 3, we need a slightly stronger definition of PCP: a *strong PCP* is a PCP for which every  $G_i$  contains an induced  $C_5$ .

We first bound the chromatic number in  $\mathcal{C}_{3,5,2k \geq 6}$  (see Lemma 3 below), which is easier because we forbid one more cycle length, and then deduce the theorem for  $\mathcal{C}_{3,2k \geq 6}$ . The proofs for  $\mathcal{C}_{3,2k \geq 6}$  and  $\mathcal{C}_{3,5,2k \geq 6}$  follow the same outline, which we informally describe here:

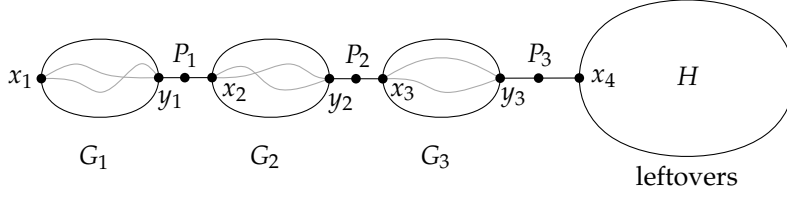


Figure 1: An informal diagram for a PCP of order 3. Grey curved lines stand for the even and odd length  $x_i y_i$ -paths.

1. If  $\chi(G)$  is large enough, then for every vertex  $v$  we can grow a PCP whose origin is  $v$  and whose leftovers are large (Lemmas 4, 5 and then Lemma 12).
2. Using (i), if  $\chi(G)$  is large enough and  $(N_0, N_1, \dots)$  is the  $v$ -levelling, we can grow a *rooted* PCP: it is a PCP in a level  $N_k$ , which has a *root*, *i.e.* a vertex in the previous level  $N_{k-1}$  whose unique neighbor in the PCP is the origin (Lemma 6 and then Lemma 13).
3. Given a rooted PCP in a level  $N_k$ , if a vertex  $x \in N_{k-1}$  has a neighbor in the last block  $H$ , then it has a neighbor in every regular block  $G_i$  (Lemma 7).
4. Given a rooted PCP of order  $\ell$  in a level  $N_k$  and a stable set  $S$  in  $N_{k-1}$ , the set of neighbors of  $S$  inside  $N_k$  can not have a big chromatic number. Consequently, the *active lift* of the PCP, defined as  $N(G_\ell) \cap N_{k-1}$ , has high chromatic number (Lemmas 8, 9 and 10 and then Lemmas 9, 14, 15).
5. The final proofs put everything together: consider a graph of  $G \in \mathcal{C}_{3,5,2k \geq 6}$  (resp.  $\mathcal{C}_{3,2k \geq 6}$ ) with chromatic number large enough. Then pick a vertex  $v$ , let  $(N_0, N_1, \dots)$  be the  $v$ -levelling and  $N_k$  be a colorful level. By (ii), grow inside  $N_k$  a rooted PCP  $P$ . Then by (iv), get an active lift  $A$  of  $P$  inside  $N_{k-1}$  with big chromatic number. Grow a rooted PCP  $P'$  inside  $A$ , and get an active lift  $A'$  of  $P'$  inside  $N_{k-2}$  with chromatic number big enough to find an edge  $xy$  (resp. a 5-hole  $C$ ) in  $A'$ . Then “clean”  $P'$  in order to get a stable set  $S$  inside the last regular block of  $P'$ , dominating this edge (resp. hole). Now find an even hole of length  $\geq 6$  in  $\{x, y\} \cup S \cup P$  (resp.  $C \cup S \cup P$ ), a contradiction.

## 2 Forbidding 5-holes

This section is devoted to the proof of the following lemma :

**Lemma 3.** *There exists a constant  $c'$  such that every graph  $G \in \mathcal{C}_{3,5,2k \geq 6}$  satisfies  $\chi(G) < c'$ .*

We follow the outline described above. Let us start with step (i):

**Lemma 4.** *Let  $G \in \mathcal{C}_{3,5,2k \geq 6}$  be a connected graph and  $v$  be any vertex of  $G$ . For every  $\delta$  such that  $\chi(G) \geq \delta \geq 18$ , there exists a PCP of order 1 with origin  $v$  and leftovers at least  $h(\delta) = \delta/2 - 8$ .*

*Proof.* The proof is illustrated on Figure 2(a). Let  $(N_0, N_1, \dots)$  be the  $v$ -levelling and  $N_k$  be a colorful level. Let  $N'_k$  be a major connected component of  $G[N_k]$ , so  $\chi(N'_k) \geq \delta/2$ . Let  $xy$  be an edge of  $N'_k$ , and  $x'$  (resp.  $y'$ ) be a neighbor of  $x$  (resp.  $y$ ) in  $N_{k-1}$ . Let  $Z' = N(\{x', y', x, y\}) \cap N'_k$  and  $Z = Z' \setminus \{x, y\}$ . Let  $z \in Z$  be a vertex having a neighbor  $z_1$  in a major connected component  $M_1$  of  $N'_k \setminus Z'$ . Observe that  $N'_k \setminus Z'$  is not empty since  $\chi(Z') \leq 6$  (the neighborhood of any vertex is a stable set). The goal is now to find two  $vz$ -paths  $P$  and  $P'$  of different parities with

interior in  $G[N_0 \cup \dots \cup \{x', y'\} \cup \{x, y\}]$ . Then we can set  $G_1 = G[P \cup P']$ ,  $P_1 = G[\{z, z_1\}]$  and  $H = G[M_1]$  as parts of the wanted PCP. In practice, we need to be a little more careful to ensure the condition on the length of  $P_1$  and the non-adjacency between  $z$  and  $H$ , which is described after finding such a  $P$  and a  $P'$ .

Let  $P_0$  (resp.  $P'_0$ ) be a  $vx'$ -path (resp.  $vy'$ -path) of length  $k - 1$  (with exactly one vertex in each level). By definition of  $Z$ ,  $z$  is connected to  $\{x', y', x, y\}$ .

1. (see Figure 2(b)) If  $z$  is connected to  $x$  or  $y$ , say  $x$ , then  $z$  is connected neither to  $x'$  nor to  $y$ , otherwise it creates a triangle. We add the path  $x'xz$  to  $P_0$  to form  $P$ . Similarly, we add either the edge  $y'z$  if it exists, or else the path  $y'yxz$  to  $P'_0$  to form  $P'$ . Observe that  $P'$  is indeed an induced path since there is no triangle.
2. (see Figure 2(c)) Otherwise,  $z$  is connected neither to  $x$  nor to  $y$ , thus  $z$  is connected to exactly one of  $x'$  and  $y'$ , since otherwise it would either create a triangle  $x', y', z$  or a 5-hole  $zx'xyy'$ , so say  $zx' \in E$  and  $zy' \notin E$ . We add the edge  $x'z$  to  $P_0$  to form  $P$ . We add the path  $y'x'z$  if  $y'x' \in E$ , otherwise add the path  $y'yx'z$  to  $P'_0$  to form  $P'$ . Observe that this is an induced path since  $G$  has no triangle and no 5-hole.

Now come the fine tuning. Choose in fact  $z_1 \in M_1 \cap N(z)$  so that  $z_1$  is connected to a major connected component  $M_2$  of  $M_1 \setminus N(z)$ . Choose  $z_2$  a neighbor of  $z_1$  in  $M_2$  such that  $z_2$  is connected to a major connected component  $M_3$  of  $M_2 \setminus N(z_1)$ . We redefine  $H = G[\{z_2 \cup M_3\}]$  and  $P_1 = G[\{z, z_1, z_2\}]$ . Then  $P_1$  is a path of length 2,  $G_1$  is colorable with 4 colors as the union of two induced paths, and  $H$  is connected. Moreover  $H$  has chromatic number at least  $\chi(N'_k) - \chi(Z') - \chi(N(z)) - \chi(N(z_1))$ . Since the neighborhood of any vertex is a stable set,  $\chi(Z') \leq 6$  and  $\chi(N(z)), \chi(N(z_1)) \leq 1$ . Thus  $\chi(H) \geq \delta/2 - 8$ .

□

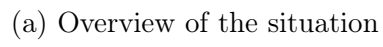
We can iterate the previous process to grow some longer PCP. In the following, for a function  $f$  and an integer  $k$ ,  $f^{(k)}$  denotes the  $k$ -th iterate of  $f$ , that is to say that  $f^{(k)}(x) = \underbrace{(f \circ \dots \circ f)}_{k \text{ times}}(x)$ .

**Lemma 5.** *Let  $h(x) = x/2 - 8$  be the function defined in Lemma 4. For every positive integers  $\ell, \delta \in \mathbb{Z}_+$ , if  $G \in \mathcal{C}_{3,5,2k \geq 6}$  is connected and satisfies  $\chi(G) \geq \delta$  and  $h^{(\ell-1)}(\delta) \geq 18$ , then from any vertex  $x_1$  of  $G$ , one can grow a PCP of order  $\ell$  with leftovers at least  $h^{(\ell)}(\delta)$ .*

*Proof.* We prove the result by induction on  $\ell$ . For  $\ell = 1$ , the result follows directly from Lemma 4. Now suppose it is true for  $\ell - 1$ , and let  $G \in \mathcal{C}_{3,5,2k \geq 6}$  be such that  $\chi(G) \geq \delta$  and  $h^{(\ell-1)}(\delta) \geq 18$ . Then  $\delta \geq h^{(\ell-1)}(\delta) \geq 18$ , so we can apply Lemma 4 to get a PCP of order 1 and leftovers at least  $h(\delta)$  from any vertex  $x_1$ . Let  $x_2$  be the common vertex between the last block  $H$  and the first path  $P_1$  of the PCP (as in the definition). Now apply the induction hypothesis to  $H$ , knowing that  $H$  is connected,  $\chi(H) \geq h(\delta) = \delta'$  and  $h^{(\ell-2)}(\delta') \geq 18$ . Then we obtain a PCP of order  $\ell - 1$  with origin  $x_2$  and leftovers at least  $h^{(\ell-2)}(\delta')$ , which finishes the proof by gluing the two PCP together. □

Now we grow the PCP in a level  $N_k$  of high chromatic number, and we want the PCP to be rooted (*i.e.* there exists a root  $u' \in N_{k-1}$  that is adjacent to the origin  $u$  of the PCP, but to no other vertex of the PCP). This is step (ii).

**Lemma 6.** *Let  $G \in \mathcal{C}_{3,5,2k \geq 6}$  be a connected graph,  $v \in V(G)$  and  $(N_0, N_1, \dots)$  be the  $v$ -levelling. Let  $h$  be the function defined in Lemma 4. For every  $k, \delta$  such that  $\chi(N_k) \geq \delta + 1$  and  $h^{(\ell-1)}(\delta) \geq 18$ , there exists a rooted PCP of order  $\ell$  in  $N_k$  with leftovers at least  $h^{(\ell)}(\delta)$ .*



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*Proof.* Let  $N'_k$  be a major connected component of  $N_k$  and  $u \in N'_k$ . Consider a neighbor  $u'$  of  $u$  in  $N_{k-1}$ . Since there is no triangle,  $N'_k \setminus N(u')$  still has big chromatic number (at least  $\delta$ ), and let  $N''_k$  be a major connected component of  $N'_k \setminus N(u')$ . Let  $z$  be a vertex of  $N(u) \cap N'_k$  having a neighbor in  $N''_k$ . Then we apply Lemma 5 in  $\{z\} \cup N''_k$  to grow a PCP of order  $\ell$  from  $z$  with leftovers at least  $h^{(\ell)}(\delta)$ . Now  $u'$  has an only neighbor  $z$  on the PCP, which is the origin.  $\square$

Let us observe the properties of such a rooted PCP. We start with step (iii):

**Lemma 7.** *Let  $v$  be a vertex of a graph  $G \in \mathcal{C}_{3,2k \geq 6}$ ,  $(N_0, N_1, \dots)$  be the  $v$ -levelling. Let  $P$  be a rooted PCP  $(G_1, P_1, \dots, G_\ell, P_\ell, H)$  of order  $\ell$  in a level  $N_k$  for some  $k$ . If  $x' \in N_{k-1}$  has a neighbor  $x$  in  $H$ , then  $x$  has a neighbor in every  $G_i$  for  $1 \leq i \leq \ell$ .*

*Proof.* Let  $u$  be the origin of the PCP and  $u'$  its root. Since  $x' \neq u'$  by definition of the root, there exists an upper  $x'u'$ -path  $P_{up}$  of length at least one. Consider a  $ux$ -path  $P$  inside the PCP. Let  $v_1, \dots, v_r$  be the neighbors of  $x'$  on this path, different from  $x$  (if any), in this order (from  $u$  to  $x$ ). Now we can show that any regular block  $G_i$  contains at least one  $v_j$ : suppose not for some index  $i$ , let  $j$  be the greatest index such that  $v_j$  is before  $x_i$ , i.e.  $v_j \in G_1 \cup P_1 \cup \dots \cup G_{i-1} \cup P_{i-1}$ .

If such an index does not exist (i.e. all the  $v_j$  are after  $G_i$ ), then there is an odd and an even path from  $u$  to  $v_1$  of length at least 3 by definition of a regular block, and this path does not contain any neighbor of  $x'$ . Close them to build two induced cycles by going through  $x'$ ,  $P_{up}$  and  $u'$ : one of them is an even cycle, and its length is at least 6.

If  $j = r$  (i.e. all the  $v_j$  are before  $G_i$ ), then we can use the same argument with a path of well-chosen parity from  $v_r$  to  $x$ , crossing  $G_i$ .

Otherwise, there is an odd and an even path in the PCP between  $v_j$  and  $v_{j+1}$ , crossing  $G_i$ , and its length is at least 4 because  $x_i$  and  $y_i$  are at distance at least 2 one from each other. We can close the even path path by going back and forth to  $x$ : this gives an even hole of length at least 6.  $\square$

Note that, in the lemma above,  $G$  is taken in  $\mathcal{C}_{3,2k \geq 6}$  and not in  $\mathcal{C}_{3,5,2k \geq 6}$ . In particular, we will use Lemma 7 in the next section as well. Let us now continue with step (iv):

**Lemma 8.** *Let  $v$  be a vertex of a graph  $G \in \mathcal{C}_{3,5,2k \geq 6}$  and  $(N_0, N_1, \dots)$  be the  $v$ -levelling. Let  $S \subseteq N_{k-1}$  be a stable set. Then  $\chi(N(S) \cap N_k) \leq 52$ .*

*Proof.* Let  $\delta = \chi(N(S) \cap N_{k-1}) - 1$ . Suppose by contradiction that  $\delta \geq 52$ , then  $h(\delta) \geq 18$  hence by Lemma 6, we can grow a rooted PCP of order 2 inside  $N(S) \cap N_k$ . Let  $u$  be the origin of the PCP and  $u'$  its root. Observe in particular that  $S$  dominates  $G_2$ . Let  $xy$  be an edge of  $G_2$ , and let  $x'$  (resp.  $y'$ ) be a neighbor of  $x$  (resp.  $y$ ) in  $S$ . By Lemma 7, both  $x'$  and  $y'$  have a neighbor in  $G_1$ . This gives an  $x'y'$ -path  $P_{down}$  with interior in  $G_1$ . In order not to create an even hole nor a 5-hole by closing it with  $x'xyy'$ , we can ensure that  $P_{down}$  is an even path of length at least 4. Moreover, there exists an upper  $x'y'$ -path  $P_{up}$ . Then either the hole formed by the concatenation of  $P_{up}$  and  $x'xyy'$ , or the one formed by the concatenation of  $P_{up}$  and  $P_{down}$  is an even hole of length  $\geq 6$ , a contradiction.  $\square$

The previous lemma allows us to prove that one can *lift* the PCP up into  $N_{k-1}$  to get a subset of vertices with high chromatic number. We state a lemma that will be reused in the next section:

**Lemma 9.** *Let  $v$  be a vertex of a graph  $G \in \mathcal{C}_{3,2k \geq 6}$  and  $(N_0, N_1, \dots)$  be the  $v$ -levelling. Let  $P$  be a rooted PCP of order  $\ell$  in a level  $N_k$  (for some  $k \geq 2$ ) with leftovers at least  $\delta$ . Let  $A = N(G_\ell) \cap N_{k-1}$  (called the active lift of the PCP). Suppose that for every stable set  $S \subseteq A$ , we have  $\chi(N(S) \cap N_k) \leq \gamma$ , then  $\chi(A) \geq \delta/\gamma$ .*

*Proof.* Let  $r = \chi(A)$ , suppose by contradiction that  $r < \delta/\gamma$  and decompose  $A$  into  $r$  stable sets  $S_1, \dots, S_r$ . Then  $N(A) \cap N_k$  is the (non-necessarily disjoint) union of  $r$  sets  $N(S_1) \cap N_k, \dots, N(S_r) \cap N_k$ , and each of them has chromatic number at most  $\gamma$  by assumption. Consequently  $\chi(N(A) \cap N_k) \leq r\gamma < \delta$  and hence  $\chi(H \setminus N(A)) \geq \chi(H) - \chi(N(A) \cap N_k) \geq 1$ . Let  $x$  be any vertex of  $H \setminus N(A)$  and  $x'$  be a neighbor of  $x$  in  $N_{k-1}$ . By construction,  $x' \notin A$  so  $x'$  has no neighbor in  $G_\ell$ . This is a contradiction with Lemma 7.  $\square$

By Lemmas 8 and 9 with  $\gamma = 52$ , we can directly deduce the following:

**Lemma 10.** *Let  $v$  be a vertex of a graph  $G \in \mathcal{C}_{3,5,2k \geq 6}$  and  $(N_0, N_1, \dots)$  be the  $v$ -levelling. Let  $P$  be a rooted PCP of order  $\ell$  in a level  $N_k$  (for some  $k \geq 2$ ) with leftovers at least  $\delta$ . Let  $A = N(G_\ell) \cap N_{k-1}$  be the active lift of the PCP, then we have  $\chi(A) \geq g(\delta) = \delta/52$ .*

We can now finish the proof, this is step (v). Recall that a sketch was provided, and it may help to understand the following proof.

*Proof of Lemma 3.* Let  $c'$  be a constant big enough so that

$$g\left(h^{(2)}\left(g\left(h^{(2)}\left(\frac{c'}{2} - 1\right)\right) - 1\right)\right) \geq 5.$$

Suppose that  $\chi(G) \geq c'$ . Pick a vertex  $v$ , let  $(N_0, N_1, \dots)$  be the  $v$ -levelling and  $N_k$  be a colorful level, so  $\chi(N_k) \geq \chi(G)/2 \geq c_1 + 1$  where  $c_1 = c'/2 - 1$ . By Lemma 6, grow a rooted PCP  $P = (G_1, P_1, G_2, P_2, H)$  inside  $N_k$  of order 2 with leftovers at least  $c_2 = h^{(2)}(c_1)$ . Then apply Lemma 10 and get an active lift  $A$  of  $P$  inside  $N_{k-1}$  with chromatic number at least  $c_3 = g(c_2)$ . Since  $h(c_3 - 1) \geq 18$ , apply again Lemma 6 to get a rooted PCP  $P' = (G'_1, P'_1, G'_2, P'_2, H')$  of order 2 inside  $N_{k-1}$  with leftovers at least  $c_4 = h^{(2)}(c_3 - 1)$ . Now apply Lemma 10 to get an active lift  $A'$  of  $P'$  inside  $N_{k-2}$  with chromatic number at least  $c_5 = g(c_4)$ .

Because of the chromatic restriction in the definition of the PCP, one can color  $G'_2$  with 4 colors. Moreover,  $G'_2$  dominates  $A'$  by definition. Thus there exists a stable set  $S \subseteq G'_2$  such that  $\chi(N(S) \cap A') \geq c_6 = c_5/4$  (since  $A'$  is the union of the  $N(S') \cap A'$  for the four stable sets  $S'$  that partition  $G'_2$ ).

Now  $c_6 > 1$  so there is an edge  $xy$  inside  $N(S) \cap A'$ . Call  $x'$  (resp.  $y'$ ) a vertex of  $S$  dominating  $x$  (resp.  $y$ ). Both  $x'$  and  $y'$  have a neighbor in  $G_2$  by definition of  $A$  and, by Lemma 7, both  $x'$  and  $y'$  also have a neighbor in  $G_1$ . This gives an  $x'y'$ -path  $P_1$  (resp.  $P_2$ ) with interior in  $G_1$  (resp.  $G_2$ ). Due to the path  $x'xyy'$  of length 3,  $P_1$  and  $P_2$  must be even paths of length at least 4. Thus the concatenation of  $P_1$  and  $P_2$  is an even hole of length at least 6, a contradiction.  $\square$

### 3 General case

This section aims at proving Theorem 2, using the result of the previous section. As already mentioned, we follow the same outline, except that we now need the existence of a  $C_5$  several times. Let us start by a technical lemma to find both an even and an odd path out of a 5-hole and its dominating set:

**Lemma 11.** *Let  $G$  be a triangle-free graph inducing a 5-hole  $C$ . Let  $S \subseteq V(G)$  be a minimal dominating set of  $C$ , assumed to be disjoint from  $C$ . If we delete the edges with both endpoints in  $S$ , then for every vertex  $t \in S$ , there exists a vertex  $t' \in S$  such that one can find an induced  $tt'$ -path of length 4 and an induced  $tt'$ -path of length 3 or 5, both with interior in  $C$ .*



*Proof.* Let  $t \in S$ , call  $v_1$  a neighbor of  $t$  on the cycle and number the others vertices of  $C$  with  $v_2, \dots, v_5$  (following the adjacency on the cycle). Since  $G$  is triangle-free,  $t$  can not be adjacent to both  $v_3$  and  $v_4$ , so up to relabeling the cycle in the other direction we assume that  $t$  is not adjacent to  $v_3$ . Let  $t' \in S$  be a vertex dominating  $v_3$ . Then  $tv_1v_2v_3t'$  is an induced path of length 4 between  $t$  and  $t'$ . Moreover,  $tv_1v_5v_4t'$  is a (non-necessarily induced) path of length 5 between  $t$  and  $t'$ . If this path is not induced, the only possible chords are  $tv_4$  and  $t'v_5$  since  $G$  is triangle-free, which in any case gives an induced  $tt'$ -path of length 3.  $\square$

Recall that in this section, we are interesting in strong PCP, *i.e.* PCP, all regular blocks  $G_i$  of which contain an induced  $C_5$ . We start with step (i):

**Lemma 12.** *Let  $c'$  be the constant of Lemma 3, let  $G \in \mathcal{C}_{3,2k \geq 6}$  and  $v$  be any vertex of  $G$ . For every  $\delta \in \mathbb{N}$  such that  $\chi(G) \geq \delta \geq 2c'$ , there exists a strong PCP of order 1 with origin  $v$  and leftovers at least  $f(\delta) = \delta/2 - 15$ .*

*Proof.* Let  $(N_0, N_1, \dots)$  be the  $v$ -levelling,  $N_k$  be a colorful level and let  $N'_k$  be a major connected component of  $G[N_k]$ , so  $\chi(N'_k) \geq c'$ . Using Lemma 3, there exists a 5-hole  $C$  in  $G[N'_k]$ . Consider a minimum dominating set  $D$  of  $C$  inside  $N_{k-1}$ .

From now on, the proof is very similar to the one of Lemma 4. Similarly, we define  $Z' = N(D \cup C) \cap N'_k$  and  $Z = Z' \setminus C$ . Let  $z \in Z$  be a vertex having a neighbor  $z_1$  in a major connected component  $M_1$  in  $N'_k \setminus Z'$ . The goal is now to find two  $vz$ -paths  $P$  and  $P'$  of different parity with interior in  $N_0 \cup \dots \cup D \cup C$ , then we can set  $G_1 = G[P \cup P' \cup C]$ ,  $P_1 = G[\{z, z_1\}]$  and  $H = G[M_1]$  as parts of the wanted PCP. In practice, we need to be a little more careful to ensure the condition on the length of  $P_1$  and the non-adjacency between  $z$  and  $H$ .

Let us now find those two paths  $P$  and  $P'$ . By definition of  $Z$ ,  $z$  also has a neighbor in  $D$  or in  $C$ .

1. If  $z$  has a neighbor  $x \in C$ , let  $y \in C$  be a vertex adjacent to  $x$  on the hole. Let  $x'$  and  $y'$  be respectively a neighbor of  $x$  and a neighbor of  $y$  in  $D$ . Observe that  $z$  is connected neither to  $x'$  nor to  $y$ , otherwise it creates a triangle. We grow  $P$  by starting from an induced path of length  $k - 1$  from  $v$  to  $x'$  and then add the path  $x'xz$ . Similarly, we grow  $P'$  by starting from an induced path of length  $k - 1$  from  $v$  to  $y'$ , and then add the edge  $y'z$  if it exists, or else the path  $y'yz$ . Observe that  $P'$  is indeed an induced path since there is no triangle.
2. If  $z$  has no neighbor in  $C$ , then it has at least one neighbor  $x'$  in  $D$ . Apply Lemma 11 to get a vertex  $y' \in D$  such that there exists an  $x'y'$ -path of length 3 or 5, and another one of length 4, both with interior in  $C$ . Observe that  $x'$  and  $y'$  cannot have a common neighbor  $u$  in  $N_{k-2} \cup \{z\}$ , otherwise there would be either a triangle  $x', u, y'$  (if  $x'y' \in E$ ), or a  $C_6$  using the  $x'y'$ -path of length 4 with interior in  $C$ . Now we grow  $P$  by starting from an induced path of length  $k - 1$  from  $v$  to  $x'$ , and add the edge  $x'z$ . We grow  $P'$  by starting from an induced path of length  $k - 1$  from  $v$  to  $y'$ , and then add the edge  $x'y'$  if it exists, otherwise add the  $x'y'$ -path of length 3 or 5 with interior in  $C$ , and then finish with the edge  $x'z$ .

Now come the fine tuning. Choose in fact  $z_1 \in M_1 \cap N(z)$  so that  $z_1$  is connected to a major connected component  $M_2$  of  $M_1 \setminus N(z)$ . Choose  $z_2$  a neighbor of  $z_1$  in  $M_2$  such that  $z_2$  is connected to a major connected component  $M_3$  of  $M_2 \setminus N(z_1)$ . We redefine  $H = G[\{z_2 \cup M_3\}]$  and  $P_1 = G[\{z, z_1, z_2\}]$ . Then  $P_1$  is a path of length 2,  $H$  is connected and  $G_1$  is colorable with 4 colors (it is easily 7-colorable as the union of a 5-hole and two paths; a careful case analysis shows that it is 4-colorable). Moreover  $H$  has chromatic number at least  $\chi(N'_k) - \chi(Z') - \chi(N(z)) - \chi(N(z_1))$ .

Since the neighborhood of any vertex is a stable set,  $\chi(Z') \leq |D| + |C| + \chi(C) \leq 13$  and  $\chi(N(z)), \chi(N(z_1)) \leq 1$ . Thus  $\chi(H) \geq \delta/2 - 15$ . □

We go on with step (ii): find a strong rooted PCP. The following lemma is proved in the same way as Lemma 6 by replacing the use of Lemma 5 by Lemma 12, so we omit the proof here.

**Lemma 13.** *Let  $G \in \mathcal{C}_{3,2k \geq 6}$  be a connected graph,  $f$  be the function defined in Lemma 12,  $v$  be a vertex of  $G$  and  $(N_0, N_1, \dots)$  be the  $v$ -levelling. For every  $k, \delta$  such that  $\chi(N_k) \geq \delta + 1 \geq 2c' + 1$ , there exists a strong rooted PCP of order 1 in  $N_k$  with leftovers at least  $f(\delta)$ .*

Step (iii) is proved by Lemma 7 from the previous section, and was valid not only for  $G \in \mathcal{C}_{3,5,2k \geq 6}$  but also for  $G \in \mathcal{C}_{3,2k \geq 6}$ . So we continue with step (iv):

**Lemma 14.** *Let  $v$  be a vertex of a graph  $G \in \mathcal{C}_{3,2k \geq 6}$ ,  $(N_0, N_1, \dots)$  be the  $v$ -levelling. Let  $S$  be a stable set inside  $N_{k-1}$ . Then  $\chi(N(S) \cap N_k) \leq 2c'$ .*

*Proof.* Suppose by contradiction that  $\chi(N(S) \cap N_k) \geq 2c' + 1$ . By Lemma 13, we can grow in  $N(S) \cap N_k$  a rooted PCP of order 1, and in particular  $S$  dominates  $G_1$ . By definition of a strong PCP, there is a 5-hole  $C$  in  $G_1$ . Since  $S$  is a dominating set of  $C$ , we can apply Lemma 11 to get two vertices  $t, t' \in S$  such that one can find both an even and an odd  $tt'$ -path with interior in  $C$  and length at least 3. Then any upper  $tt'$ -path close a hole of even length  $\geq 6$ . □

In fact, as in previous section, we can directly deduce from Lemmas 9 and 14 that one can lift the PCP up into  $N_{k-1}$  to get a subset of vertices with high chromatic number:

**Lemma 15.** *Let  $G \in \mathcal{C}_{3,2k \geq 6}$ ,  $v \in V(G)$  and  $(N_0, N_1, \dots)$  be the  $v$ -levelling. Let  $P$  be a strong rooted PCP of order 1 in a level  $N_k$  (for some  $k \geq 2$ ) with leftovers  $\delta$ . Let  $A = N(G_1) \cap N_{k-1}$  be the active lift of the PCP. If  $\delta \geq 2c'$ , then  $\chi(A) \geq \varphi(\delta) = \frac{\delta}{2c'}$ .*

We are now ready to finish the proof, this is step (v). Recall that a sketch was given and may be useful to have a less technical overview of the proof.

*Proof of Theorem 2.* Let  $c$  be a constant such that

$$\varphi \left( f \left( \varphi \left( f \left( \frac{c}{2} - 1 \right) \right) - 1 \right) \right) \geq 4c'.$$

Suppose that  $G \in \mathcal{C}_{3,2k \geq 6}$  has chromatic number  $\chi(G) \geq c$ . Then pick a vertex  $v$ , let  $(N_0, N_1, \dots)$  be the  $v$ -levelling and  $N_k$  be a colorful level, consequently  $\chi(N_k) \geq c_1 + 1 = c/2$ . Apply Lemma 13 and grow inside  $N_k$  a strong rooted PCP  $P = (G_1, P_1, H)$  of order 1 with leftovers at least  $c_2 = f(c_1)$ . Then apply Lemma 15 and get an active lift  $A = N(G_1)$  of  $P$  inside  $N_{k-1}$  with chromatic number at least  $c_3 = \varphi(c_2)$ . By Lemma 13, we can obtain a strong rooted PCP  $P' = (G'_1, P'_1, H')$  inside  $A$  with leftovers at least  $c_4 = f(c_3 - 1)$ , and by Lemma 15 we obtain an active lift  $A'$  of  $P'$  inside  $N_{k-2}$  with chromatic number at least  $c_5 = \varphi(c_4)$ . Because of the chromatic restriction in the definition of the PCP, one can color  $G'_1$  with 4 colors. Moreover,  $G'_1$  dominates  $A'$  by definition. Thus there exists a stable set  $S \subseteq P'$  such that  $\chi(N(S) \cap A') \geq c_6 = c_5/4$ . Now  $c_6 \geq c'$  thus Lemma 3 proves the existence of a 5-hole  $C$  inside  $N(S) \cap A'$ . Let us give an overview of the situation: we have a 5-hole  $C$  inside  $N_{k-2}$ , dominated by a stable set  $S$  inside  $N_{k-1}$ , and every pair of vertices  $t, t'$  of  $S$  can be linked by a  $tt'$ -path  $P_{down}$  with interior in  $G_1 \subseteq N_k$ . Lemma 11 gives the existence of two vertices  $t, t' \in S$  linked by both an odd path and an even path of length  $\geq 3$  with interior in  $C$ . Closing one of these paths with  $P_{down}$  gives an induced even hole of length  $\geq 6$ , a contradiction. □

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